

Tilting modules over duplicated algebras

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Abstract Let A be a finite dimensional hereditary algebra over a field k and $A^{(1)}$ the duplicated algebra of A . We first show that the global dimension of endomorphism ring of tilting modules of $A^{(1)}$ is at most 3. Then we investigate embedding tilting quiver $\mathcal{K}(A)$ of A into tilting quiver $\mathcal{K}(A^{(1)})$ of $A^{(1)}$. As applications, we give new proofs for some results of D.Happel and L.Unger, and prove that every connected component in $\mathcal{K}(A)$ has finite non-saturated points if A is tame type, which gives a partially positive answer to the conjecture of D.Happel and L.Unger in [10]. Finally, we also prove that the number of arrows in $\mathcal{K}(A)$ is a constant which does not depend on the orientation of Q if Q is Dynkin type.

1 Introduction

Tilting theory usually has two aspects. One is the external aspect, which is used to compare Λ -mod to $\text{End}_\Lambda T$ -mod for a tilting Λ module T . The other internal aspect, which is to study tilting modules for a fixed algebra Λ and to try to gather information about Λ -mod, see [6, 7, 8, 9, 10] for more details. Recently, tilting theory has remarkable development in cluster categories, which was introduced in [5]. Now, cluster categories become a successful model for acyclic cluster algebras, this new discovery has rapidly promoted research on this direction.

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According to [1], we know that tilting modules of duplicated algebra $A^{(1)}$ of hereditary algebra A have strong relationship with cluster tilting objects in cluster category \mathcal{C}_A . For example, there is a one-to-one correspondence between basic tilting $A^{(1)}$ -modules with projective dimension at most one and basic cluster tilting objects in \mathcal{C}_A .

It is well known that the tilting quiver $\mathcal{K}(A)$ of a hereditary algebra A usually is not connected. For example, the tilting quiver of Kronecker algebra consists of two connected components. However, according to [14], we know that the tilting quiver $\mathcal{K}(A^{(1)})$ of A is connected. This motivates further investigation on the structure of tilting modules of hereditary algebras and of tilting modules of duplicated algebras with projective dimension at most one.

In this paper, we focus on the structure properties of tilting modules with projective dimension at most one for duplicated algebra $A^{(1)}$, and prove that the global dimension of endomorphism ring of this kinds tilting modules is at most 3 (see Theorem 3.1 in Section 3). In Section 4, we are interested in the relationship between the tilting quivers $\mathcal{K}(A)$ and $\mathcal{K}(A^{(1)})$, and prove some embedding theorems.

In Section 5, we give new proofs for some results of D.Happel and L.Unger by using embedding theorem, and prove that every connected component in $\mathcal{K}(A)$ has finite non-saturated points if A is tame type, which gives a partially positive answer to the conjecture of D.Happel and L.Unger in [10]. We also prove that the number of arrows in $\mathcal{K}(A)$ is a constant which does not depend on the orientation of Q if Q is Dynkin type. We fix notations and recall some facts needed for our later use in Section 2.

2 preliminaries

Let Λ be a finite dimensional k -algebra over a field k and $\Lambda\text{-mod}$ be the category of all finitely generated left Λ -modules. We denote by $\Lambda\text{-ind}$ the full subcategory of $\Lambda\text{-mod}$ consisting of indecomposable Λ modules, and denote by $\text{pd}_\Lambda X$ the projective dimension of an Λ module X and by $\text{gl.dim } \Lambda$ the global dimension of Λ . Let

$D = \text{Hom}_k(-, k)$ be the standard duality between $\Lambda\text{-mod}$ and $\Lambda^{\text{op}}\text{-mod}$, and τ_Λ be the Auslander-Reiten translation of Λ . The Auslander-Reiten quiver of Λ is denoted by Γ_Λ .

Given any module $M \in \Lambda\text{-mod}$, we denote by M^\perp the subcategory of $\Lambda\text{-mod}$ with objects $X \in \Lambda\text{-mod}$ satisfying $\text{Ext}_\Lambda^i(M, X) = 0$ for all $i \geq 1$ and by ${}^\perp M$ the subcategory of $\Lambda\text{-mod}$ with objects $X \in \Lambda\text{-mod}$ satisfying $\text{Ext}_\Lambda^i(X, M) = 0$ for $i \geq 1$. We denote by $\Omega_\Lambda^i M$ and $\Omega_\Lambda^{-i} M$ the i -th syzygy and cosyzygy of M respectively, and denote by $\text{gen } M$ the subcategory of $\Lambda\text{-mod}$ whose objects are generated by M . We may decompose M as $M \cong \bigoplus_{i=1}^m M_i^{d_i}$, where each M_i is indecomposable, $d_i > 0$ for each i , and M_i is not isomorphic to M_j if $i \neq j$. The module M is called basic if $d_i = 1$ for any i . The number of non-isomorphic indecomposable modules occurring in the direct sum decomposition above is uniquely determined and it is denoted by $\delta(M)$. The full subcategory having as objects the direct sums of indecomposable summands of M is denoted by $\text{add } M$.

A module $T \in \Lambda\text{-mod}$ is called a tilting module if the following conditions are satisfied:

- (1) $\text{pd}_\Lambda T \leq 1$;
- (2) $\text{Ext}_\Lambda^1(T, T) = 0$;
- (3) There is an exact sequence $0 \longrightarrow \Lambda \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$ with $T_i \in \text{add } T$ for $0 \leq i \leq 1$.

An Λ module M satisfying the conditions (1) and (2) of the definition above is called a partial tilting module and if moreover $\delta(M) = \delta(\Lambda) - 1$, then M is called an almost complete tilting module. Let M be a partial tilting module and X be an Λ -module such that $M \oplus X$ is a tilting module and $\text{add } M \cap \text{add } X = 0$. Then X is called a complement to M .

Let T be a tilting Λ module and $B = \text{End}_\Lambda T$. Then $(\mathcal{T}(T), \mathcal{F}(T))$ is the torsion pair in $\Lambda\text{-mod}$ generated by T , where $\mathcal{T}(T) = T^\perp = \text{gen } T$ and $\mathcal{F}(T) = \{X \in \Lambda\text{-mod} \mid \text{Hom}_\Lambda(T, X) = 0\}$, the corresponding torsion pair in $B\text{-mod}$ is $(\mathcal{X}(T), \mathcal{Y}(T))$, where $\mathcal{X}(T) = \{X \in B\text{-mod} \mid T \otimes_B X = 0\}$ and $\mathcal{Y}(T) = \{Y \in B\text{-mod} \mid \text{Tor}_1^B(T, Y) = 0\}$.

Lemma 2.1. Take the notations as above. Then

(1) $\text{Hom}_\Lambda(T, -) : \mathcal{T}(T) \longrightarrow \mathcal{Y}(T)$ is an equivalence functor;

(2) Let $M \in \mathcal{T}(T)$. Then $\text{pd}_B \text{Hom}_\Lambda(T, M) \leq \text{pd}_\Lambda M$. \square

Let \mathcal{T}_Λ be the set of all basic tilting Λ modules up to isomorphism. According to [10], we define the tilting quiver $\mathcal{K}(\Lambda)$ of Λ as the following. The vertices of $\mathcal{K}(\Lambda)$ are the elements of \mathcal{T}_Λ . There is an arrow $T' \rightarrow T$ in $\mathcal{K}(\Lambda)$ if and only if $T' = M \oplus X$ and $T = M \oplus Y$ with X and Y indecomposable such that there is a short exact sequence $0 \rightarrow X \rightarrow E \rightarrow Y \rightarrow 0$ with $E \in \text{add } M$.

Let \mathcal{C} be a full subcategory of $\Lambda\text{-mod}$, $C_M \in \mathcal{C}$ and $\varphi : C_M \longrightarrow M$ with $M \in A\text{-mod}$. Recall from [3], the morphism φ is a right \mathcal{C} -approximation of M if the induced morphism $\text{Hom}_\Lambda(C, C_M) \longrightarrow \text{Hom}_\Lambda(C, M)$ is surjective for any $C \in \mathcal{C}$. A minimal right \mathcal{C} -approximation of M is a right \mathcal{C} -approximation which is also a right minimal morphism, i.e., its restriction to any nonzero summand is nonzero. The subcategory \mathcal{C} is called contravariantly finite if any module $M \in A\text{-mod}$ admits a (minimal) right \mathcal{C} -approximation. The notions of (minimal) left \mathcal{C} -approximation and of covariantly finite subcategory can be defined dually. It is well known that $\text{add } M$ is both a contravariantly finite subcategory and a covariantly finite subcategory.

Let M, N be two indecomposable Λ -modules. A path from M to N in $\Lambda\text{-ind}$ is a sequence of non-zero morphisms $M = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} M_t = N$ with all M_i in $\Lambda\text{-ind}$. Following [13], we denote by $M \leq N$ the existence of such a path, and we say that M is a predecessor of N (or that N is a successor of M).

From now on, let $A = kQ$ be a finite dimensional hereditary algebra over a field k , and let $Q_0 = \{1, \dots, n\}$ be the vertexes set of Q . Recall from [1], $A^{(1)} = \begin{pmatrix} A & 0 \\ DA & A \end{pmatrix}$ is said to be the duplicated algebra of A . We know that $A^{(1)}$ contains

two copies of A given by $eA^{(1)}e$ and by $e'A^{(1)}e'$ respectively, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

and $e' = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. We denote the first one by A and the second one by A' .

Then we denote by Q' the quiver of A' , by i' the vertex of Q'_0 corresponding to $i \in Q_0$, and by e'_i the corresponding idempotent. Let S_x, P_x, I_x denote respectively the corresponding simple, indecomposable projective and indecomposable injective module in $A^{(1)}$ corresponding to $x \in (Q_0 \cup Q'_0)$. Note that A -ind can be embedded in $A^{(1)}$ -ind, and $P_{x'}$ is an indecomposable projective-injective $A^{(1)}$ module for every $x' \in Q'_0$.

We denote by Σ_0 the set of all non-isomorphic indecomposable projective A -modules and by Σ_i the set of $\Omega_{A^{(1)}}^{-i} \Sigma_0$. Note that $2 \leq \text{gl.dim } A^{(1)} \leq 3$. Moreover, if A is representation-infinite, then $\text{gl.dim } A^{(1)} = 3$. (See [12])

Let $\mathcal{L}_{A^{(1)}}$ be the left part of $\text{mod } A^{(1)}$. By definitions, $\mathcal{L}_{A^{(1)}}$ is the full subcategory of $\text{mod } A^{(1)}$ consisting of all indecomposable $A^{(1)}$ -modules such that if L is a predecessor of M , then the projective dimension $\text{pd } L$ of L is at most one.

The following result is proved in [15] and will be used in our further research.

Lemma 2.2. Let $A = kQ$ be a finite dimensional hereditary algebra over a field k and $A^{(1)}$ be the duplicated algebra of A . Then the tilting quiver $\mathcal{K}(A^{(1)})$ is connected. \square

Remark. We should mention that $A^{(1)}$ was generalized to m -replicated algebra $A^{(m)}$ for any integer $m \geq 1$ in [2], and this kind of algebras has been proved having closely relationship with m -cluster categories, and was extensively investigated in [11, 12, 15, 16].

Throughout this paper, we follow the standard terminology and notations used in the representation theory of algebras as in [4, 13].

3 Global dimension of endomorphism algebras of tilting modules

Let $A = kQ$ be a finite dimensional hereditary algebra over a field k and $A^{(1)}$ be the duplicated algebra of A . For convenience, we denote by $\bar{P}_{A^{(1)}}$ the direct sum of indecomposable projective-injective $A^{(1)}$ modules. In this section, we prove that global dimension of the endomorphism algebras of tilting $A^{(1)}$ modules is at most 3.

Let $T \oplus \bar{P}_{A^{(1)}}$ be a basic tilting $A^{(1)}$ module and $B = \text{End}(T \oplus \bar{P}_{A^{(1)}})$. We know that $T \in \text{add } \mathcal{L}_{A^{(1)}}$ and $\delta(T) = \delta(A) = n$.

By [1] we know that $\text{gl.dim } A^{(1)} \leq 3$ and $\text{gl.dim } A^{(1)} = 3$ if A is representation infinite. It is well known that $\text{gl.dim } A^{(1)} - 1 \leq \text{gl.dim } B \leq \text{gl.dim } A^{(1)} + 1$ which implies that $\text{gl.dim } B \leq 4$. However, we can prove the following surprising result.

Theorem 3.1. *Take the notations as above. Then $\text{gl.dim } B \leq 3$.*

Proof Let $T = \bigoplus_{i=1}^n T_i$ and $\bar{P}_{A^{(1)}} = \bigoplus_{i=1}^n \bar{P}_{i'}$. Let S be a simple B module.

Case 1. Assume that S is the top of $\text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, T_i)$. Then we have an exact sequence $0 \rightarrow Y \rightarrow \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, T_i) \rightarrow S \rightarrow 0$. Note that Y lies in $\mathcal{Y}(T \oplus \bar{P}_{A^{(1)}})$ since $\mathcal{Y}(T \oplus \bar{P}_{A^{(1)}})$ is a torsion free class and $\text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, T_i)$ lies in $\mathcal{Y}(T \oplus \bar{P}_{A^{(1)}})$. According to Lemma 1.1, there exists $M \in \mathcal{T}(T \oplus \bar{P}_{A^{(1)}})$ such that $Y = \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, M)$, hence M is a predecessor of T_i and $M \in \mathcal{L}_{A^{(1)}}$ since T_i lies in $\mathcal{L}_{A^{(1)}}$. Therefor $\text{pd}_{A^{(1)}} M \leq 1$ and by Lemma 2.1 again, we know that $\text{pd}_B Y = \text{pd}_B \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, M) \leq \text{pd}_{A^{(1)}} M \leq 1$, which implies that $\text{pd}_B S \leq 2$.

Case 2. Let S be the top of $\text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, \bar{P}_{i'})$. Then we have an exact sequence $0 \rightarrow Y \rightarrow \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, \bar{P}_{i'}) \rightarrow S \rightarrow 0$. By using the same argument as in Case 1, we know that $Y = \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, M)$ with $M \in \mathcal{T}(T \oplus \bar{P}_{A^{(1)}})$ such that M is a predecessor of Σ_2 , hence $\text{pd}_{A^{(1)}} M \leq 2$. According to Lemma 2.1, we know that $\text{pd}_B Y = \text{pd}_B \text{Hom}_{A^{(1)}}(T \oplus \bar{P}_{A^{(1)}}, M) \leq \text{pd}_{A^{(1)}} M \leq 2$, which implies that $\text{pd}_B S \leq 3$. This proves that $\text{gl.dim } B \leq 3$. \square

4 Embedding of the tilting quiver

Let $A = kQ$ be a finite dimensional hereditary algebra over a field k and $A^{(1)}$ be the duplicated algebra of A . In this section, we investigate the relationship between the tilting quivers of A and of $A^{(1)}$.

Theorem 4.1. *Let $\mathcal{K}(A)$ (resp. $\mathcal{K}(A^{(1)})$) be the tilting quiver of A (resp. $A^{(1)}$). Then there is an arrow $T' \rightarrow T$ in $\mathcal{K}(A)$ if and only if $T' \oplus \bar{P}_{A^{(1)}} \rightarrow T \oplus \bar{P}_{A^{(1)}}$ is an arrow in $\mathcal{K}(A^{(1)})$.*

Proof Let T is a tilting A module. It is easy to see that $T \oplus \bar{P}_{A'}$ is a tilting $A^{(1)}$ module.

Assume that $T' \rightarrow T$ is an arrow in $\mathcal{K}(A)$, then there is an almost tilting A module M such that $T' = M \oplus X$ and $T = M \oplus Y$ with X and Y are indecomposable. Moreover, there is an exact sequence $0 \rightarrow X \xrightarrow{f} E \xrightarrow{g} Y \rightarrow 0$ is an exact sequence with $E \in \text{add } M$, such that f is a left minimal add M -approximation and that g is a right minimal add M -approximation.

It follows that $T' \oplus \bar{P}_{A^{(1)}} = M \oplus X \oplus \bar{P}_{A^{(1)}}$ and $T \oplus \bar{P}_{A^{(1)}} = M \oplus Y \oplus \bar{P}_{A^{(1)}}$ are tilting $A^{(1)}$ modules, and g, f are also minimal add $M \oplus \bar{P}_{A^{(1)}}$ -approximation, since $\bar{P}_{A^{(1)}}$ is a projective-injective module. Hence $T' \oplus \bar{P}_{A^{(1)}} \rightarrow T \oplus \bar{P}_{A^{(1)}}$ is an arrow in $\mathcal{K}(A^{(1)})$.

The converse can be proved similarly. This completes the proof. \square

Theorem 4.2. *Each point in $\mathcal{K}(A^{(1)})$ has n arrows connected.*

Proof Let $T \oplus \bar{P}_{A^{(1)}}$ be a basic tilting $A^{(1)}$ module. Then $\delta(T) = \delta(P_{A^{(1)}}) = n$. Assume that $T = \bigoplus_{i=1}^n T_i$, and let $T[i] = \bigoplus_{j \neq i} T_j$. Then $T[i] \oplus \bar{P}_{A^{(1)}}$ is an almost tilting $A^{(1)}$ module.

According to [14], we know that $T[i] \oplus \bar{P}_{A^{(1)}}$ has exactly two non-isomorphic complements with projective dimension at most 1, and one of them is T_i .

Note that if T_i is the source complement, then there exists an arrow $T \oplus \bar{P}_{A^{(1)}} \rightarrow *$. Otherwise, there is an arrow $* \rightarrow T \oplus \bar{P}_{A^{(1)}}$. This implies that there are exactly n arrows connected with $T \oplus \bar{P}_{A^{(1)}}$. The proof is completed. \square

Theorem 4.3. *Let M be a basic almost tilting A module. Then $(\mathbf{dim} M)_i = 0$ if and only if $M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}$ is a tilting $A^{(1)}$ module.*

Proof Note that $\text{pd}_{A^{(1)}} M \leq 1$ and $\tau_{A^{(1)}}^{-1} I_i \in \mathcal{L}_{A^{(1)}}$, it follows that

$$\text{pd}_{A^{(1)}} (M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}) \leq 1.$$

We have that

$$\text{Ext}_{A^{(1)}}^1(M, \tau_{A^{(1)}}^{-1} I_i) \cong D\text{Hom}_{A^{(1)}}(\tau_{A^{(1)}}^{-1} I_i, \tau_{A^{(1)}} M) = 0,$$

and that

$$\text{Ext}_{A^{(1)}}^1(\tau_{A^{(1)}}^{-1} I_i, M) \cong D\text{Hom}_{A^{(1)}}(M, I_i) = 0,$$

hence

$$\text{Ext}_{A^{(1)}}^1(M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}, M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}) = 0,$$

then $M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}$ is a tilting $A^{(1)}$ module, since

$$\delta(M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}) = \delta(A^{(1)}) = 2n.$$

Conversely, if $M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \bar{P}_{A^{(1)}}$ is a tilting $A^{(1)}$ module, then

$$\text{Ext}_{A^{(1)}}^1(\tau_{A^{(1)}}^{-1} I_i, M) \cong D\text{Hom}_{A^{(1)}}(M, I_i) = 0,$$

this implies that $\text{Hom}_A(M, I_i) = 0$ and $(\mathbf{dim} M)_i = 0$. The proof is completed. \square

The following corollary can be proved easily.

Corollary 4.4. *Let M be an almost tilting A module and M is not sincere, then the dimension vector $\mathbf{dim} M$ of M has exactly one component equals to 0.*

Proof Assume by contrary that there are two or more different components of $\mathbf{dim} M$ equal to zero. That is, there are $i \neq j$ such that $(\mathbf{dim} M)_i = (\mathbf{dim} M)_j = 0$. By using the method in the proof of Theorem 4.3, we know that $M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \tau_{A^{(1)}}^{-1} I_j \oplus \bar{P}_{A^{(1)'}}$ is a tilting $A^{(1)}$ module, then $\delta(M \oplus \tau_{A^{(1)}}^{-1} I_i \oplus \tau_{A^{(1)}}^{-1} I_j \oplus \bar{P}_{A^{(1)'}}) = 2n + 1$, which is a contradiction. \square

5 Applications of the embedding theorem

Let $A = kQ$ be a finite dimensional hereditary algebra over a field k and $A^{(1)}$ be the duplicated algebra of A . In this section, we give new proofs for some results of D.Happel and L.Unger by using embedding theorem, and obtain a partially positive answer to the conjecture of D.Happel and L.Unger in [10], which says that every connected component in $\mathcal{K}(A)$ has finite non-saturated points. We also prove that the number of arrows in $\mathcal{K}(A)$ is a constant which does not depend on the orientation of Q if Q is Dynkin type.

The following proposition is the main result of [7], we give a new proof by using embedding theorem in Section 3.

Proposition 5.1.^[7] *Let M be an almost tilting A module. If M is sincere, then M has two non-isomorphic indecomposable complements, and if M is non-sincere, then M has exactly one complement.*

Proof Note that $M \oplus \bar{P}_{A^{(1)}}$ is an almost tilting $A^{(1)}$ module, according to [15], we know that $M \oplus \bar{P}_{A^{(1)}}$ has two non-isomorphic indecomposable complements $X, Y \in \text{ind } A \cup \{\tau_{A^{(1)}}^{-1} I_x | x \in Q\}$.

If M is sincere, then $X, Y \in \text{ind } A$. Otherwise, we may assume that $X = \tau_{A^{(1)}}^{-1} I_i$, according to Theorem 4.3, $(\dim M)_i = 0$ which means M not sincere.

If M is non-sincere, by Corollary 4.4 we know that $M \oplus \bar{P}_{A^{(1)}}$ has exactly one complement looking like $\tau_{A^{(1)}}^{-1} I_i$, and the other complement must lie in $\text{ind } A$ which is also the only complement for the almost tilting A modules M . \square

Recall from [10], let $T \in \mathcal{K}(A)$. We denote by $s(T)$ (resp. $e(T)$) the number of arrows starting (resp. ending) at T in $\mathcal{K}(A)$, then $\sigma(T) = s(T) + e(T) \leq \delta(A) = n$. We say that T is saturated if $\sigma(T) = n$. The following result is stated as Proposition 3.2 in [10], and we provide a new proof here.

Proposition 5.2.^[10] *Let T be a basic tilting A module, then the point T in the tilting quiver $\mathcal{K}(A)$ of A is saturated if and only if $(\dim T)_i \geq 2, \forall i \in Q_0$.*

Proof Assume that T is saturated and there is some $i \in Q_0$ with $(\mathbf{dim} T)_i = 1$, then there must be an indecomposable summand T_k of T such that $(\mathbf{dim} T_k)_i = 1$. So $T[k]$ is non-sincere since the i^{th} component of $\mathbf{dim} T[k]$ is 0. According to Proposition 4.1, there is only one complement for $T[k]$ in $A\text{-mod}$. This means that T is not saturated, and we get a contradiction.

Conversely, If $(\mathbf{dim} T)_i \geq 2$ for all $i \in Q_0$ and T is not saturated, then we know that there exists at least one $T[k]$, in $\text{mod } A$, which has the unique complement T_k , hence $T[k]$ is non-sincere. We may assume that $(\mathbf{dim} T[k])_i = 0$, according to Theorem 4.3, $T[k] \oplus \bar{P}_{A(1)}$ has a complement $\tau_{A(1)}^{-1} I_j$ in $\text{mod } A^{(1)}$. It follows that $T[k] \oplus \bar{P}_{A(1)}$ has two complements $X = T_k$ and $\tau_{A(1)}^{-1} I_j$, which means that there is an exact sequence $0 \rightarrow X \rightarrow E \rightarrow \tau_{A(1)}^{-1} I_j \rightarrow 0$ with $E \in \text{add}(T[k] \oplus \bar{P}_{A(1)})$. Applying $\text{Hom}_{A(1)}(-, I_j)$ we obtain the following exact sequence

$$\text{Hom}_{A(1)}(E, I_j) \rightarrow \text{Hom}_{A(1)}(X, I_j) \rightarrow \text{Ext}_{A(1)}^1(\tau_{A(1)}^{-1} I_j, I_j) \rightarrow 0.$$

$\text{Hom}_{A(1)}(E, I_j) = 0$ since $\text{Hom}_{A(1)}(T[k], I_j) = 0$ and $\text{Hom}_{A(1)}(\bar{P}_{A(1)}, I_j) = 0$, hence

$$(\mathbf{dim} X)_j = \dim \text{Hom}_{A(1)}(X, I_j) = \dim \text{Ext}_{A(1)}^1(\tau_{A(1)}^{-1} I_j, I_j) = \dim D\text{Hom}(I_j, I_j) = 1.$$

It follows that $(\mathbf{dim} T)_j = (\mathbf{dim} T[k])_j + (\mathbf{dim} X)_j = 1$, which contradicts with the assumption. \square

Corollary 5.3.^[10] *Let $A = kQ$ be a finite dimensional hereditary algebra over a field k . Then A and DA are not saturated in the tilting quiver $\mathcal{K}(A)$.*

Proof Let i be a source vertex of Q_0 . Then $(\mathbf{dim} \bigoplus_{j \in Q_0, j \neq i} P_j)_i = 0$, hence A is not saturated. That DA is not saturated can be proved dually. \square

We give a very different proof for Theorem 3.5 in [10] as following.

Proposition 5.4.^[10] *Let $A = kQ$ be a finite dimensional hereditary algebra over a field k . Then each connected component in the tilting quiver $\mathcal{K}(A)$ has a non-saturated point.*

Proof If $\mathcal{K}(A)$ is connected, it is easy to see that A is one of non-saturated point in $\mathcal{K}(A)$. Now, we assume that $\mathcal{K}(A)$ is not connected. If $\mathcal{K}(A)$ has one

component such that every point is saturated, according to Proposition 5.1 and Proposition 4.2, $\mathcal{K}(A)$ can be embedded into $\mathcal{K}(A^{(1)})$ and the only change is that every basic tilting A module T is replaced by $T \oplus \bar{P}_{A^{(1)}}$ and the arrows keep no changes. This implies that the component, which every point is saturated, is isolated. In particular, $\mathcal{K}(A^{(1)})$ has at least two components, which is contradict with Lemma 2.2. This completes the proof. \square

Let $A = kQ$ be a finite dimensional hereditary algebra over a field k . D.Happel and L.Unger in [10] conjectured that every connected component in $\mathcal{K}(A)$ has finite non-saturated points. The following theorem gives a partially positive answer to this conjecture.

Theorem 5.5. *Let $A = kQ$ be a finite dimensional hereditary algebra over a field k . If Q is either Dynkin or Euclidean type, then every connected component of $\mathcal{K}(A)$ has finite non-saturated points.*

Proof If A is a Dynkin type, then $\mathcal{K}(A)$ is a finite quiver and our consequence is true. Now we assume that Q is an Euclidean type. Let T be a non-saturated point in $\mathcal{K}(A)$. Then $\mathbf{dim} T$ has at least one component equal to 1. We denote by Δ the set of non-saturated points in $\mathcal{K}(A)$, and we divide Δ into different parts and put $\Delta_i = \{T \in \Delta \mid (\mathbf{dim} T)_i = 1\}$.

We claim that Δ_i is a finite set for $1 \leq i \leq n$. In fact, $\forall T \in \Delta_i$, we know that $(\mathbf{dim} T)_i = 1$. Let $T = \bigoplus_{i=1}^n T_i$. Then there is a T_k with $(\mathbf{dim} T_k)_i = 1$. Let $Q_{(i)}$ be the quiver by removing the vertex i from Q_0 and removing all the arrows connected with i . Then $T[k]$ can be regarded as a basic tilting $kQ_{(i)}$ module, and $kQ_{(i)}$ is representation-finite, hence Δ_i is a finite set, it follows that $\Delta = \bigcup_{i=1}^n \Delta_i$ is also a finite set. This completes the proof. \square

Theorem 5.6. *Let $A = kQ$ and Q be Dynkin type. Then the number of arrows in $\mathcal{K}(A)$ is a constant and does not depend on the orientation of Q .*

Proof According to Theorem 4.1, we know that $\mathcal{K}(A)$ can be embedded into $\widehat{\mathcal{K}(A^{(1)})}$. We denote by $\widehat{\mathcal{K}(A)}$ the full subquiver of $\mathcal{K}(A)$ in $\widehat{\mathcal{K}(A^{(1)})}$. Note that $\widehat{\mathcal{K}(A)}$ has the same vertices as $\mathcal{K}(A)$, and every vertex in $\widehat{\mathcal{K}(A)}$ connected with

n arrows. Let s be the number of basic tilting A modules, it is well known that s is a fixed number which is independent of the orientation of Q .

Let $Q_{\hat{x}} = Q \setminus \{x\}$ be the quiver obtained from Q with a vertex $x \in Q_0$ removed. Then $kQ_{\hat{x}}$ is a representation finite hereditary algebra. We denote by m_x the number of basic tilting $kQ_{\hat{x}}$ modules, then m_x is a fixed number which does not depend on the orientation of Q .

Let m be the number of arrows should be added in order to get $\widehat{\mathcal{K}(A)}$ from $\mathcal{K}(A)$. Note that every tilting $kQ_{\hat{x}}$ module is a non-sincere almost tilting A module, and there is one corresponding arrow in $\widehat{\mathcal{K}(A)} \setminus \mathcal{K}(A)$. On the other hand, every arrow in $\widehat{\mathcal{K}(A)} \setminus \mathcal{K}(A)$ corresponding to one almost tilting A module which can be seen as tilting $kQ_{\hat{x}}$ module for some $x \in Q_0$. According to Theorem 4.3, $m = \sum_{x \in Q_0} m_x$ is a fixed number.

Let t be the number of arrows in $\mathcal{K}(A)$. According to Theorem 4.2, we have an equation $2t + m = ns$, hence $t = \frac{ns-m}{2}$ is a fixed number which does not depend on the orientation of Q , that is, t is a constant. The proof is finished. \square

Remark. Theorem 5.6 is more general than the result in [14] which stands only for A_n and D_n type, and our proof is different and simpler.

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